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On the asymptotic expansion of the Kashaev invariant and the twisted Reidemeister torsion of two-bridge knots

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1 Introduction

This note is a survey of the joint work [8] with Tomotada Ohtsuki.

In [2, 3], Kashaev defined the Kashaev invariant $\langle L\rangle_N \in \mathbb{C}$ of a link L for $N=2,3,\cdots$ by using the quantum dilogarithm at $q=e^{2\pi\sqrt{-1}/N}$. In [4], he conjectured that, for any hyperbolic link L, $\frac{2\pi}{N}\log|\langle L\rangle_N|$ goes to the hyperbolic volume of S^3-L as $N\to\infty$. In [6], Ohtsuki proposed the following refined conjecture:

Conjecture 1 ([6]). *For any hyperbolic knot K, the asymptotic expansions of the Kashaev invariant of K is presented by the following form,*

$$\langle K\rangle_N = e^{N\varsigma(K)}N^{3/2}\,\omega(K)\cdot\Big(1+\sum_{i=1}^{d}\kappa_i(K)\cdot\big(\frac{2\pi\sqrt{-1}}{N}\big)^i + O\big(\frac{1}{N^{d+1}}\big)\Big), \tag{1}$$

for any d, where $\omega(K)$ and $\kappa_i(K)$'s are some scalars only depending on K. Here $\varsigma(K) = \frac{1}{2\pi\sqrt{-1}}\big(\mathrm{cs}(S^3-K)+\sqrt{-1}\,\mathrm{vol}(S^3-K)\big)$, where "cs" and "vol" denote the Chern-Simons invariant and the hyperbolic volume.

It is shown in [6, 9, 7] that, for any hyperbolic knot K with up to 7 crossings, Conjecture 1 holds. Moreover, the following is conjectured for $\omega(K)$ of (1):

Conjecture 2. *For any hyperbolic knot K,*

$$2\pi\sqrt{-1}\,\omega(K)^2 = \pm\tau(K),$$

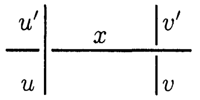
where $\tau(K)$ is the twisted Reidemeister torsion associated with the holonomy representation of the hyperbolic structure of the complement of K.

For the figure-eight knot, this conjecture was shown by Andersen and Hansen [1] and H. Murakami [5]. We show

Theorem 1 ([8]). *For any hyperbolic knot K with up to 7 crossings, Conjecture 2 holds.*

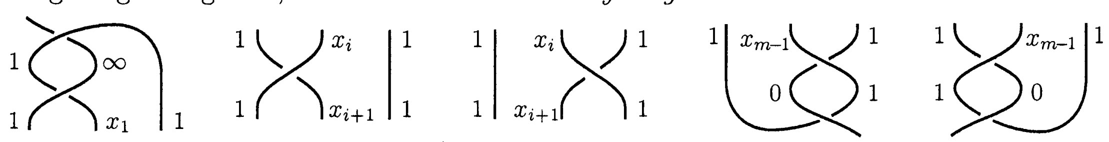
2 Results

Let us review a parameterized knot diagram of an open knot, where an open knot is a 1-tangle whose closure is a knot. We parameterize edges of an open knot diagram by parameters in $\mathbb{C} \cup \{\infty\}$. We parameterize edges adjacent to unbounded regions by 1. We parameterize edges next to the terminal edges by 0 or ∞; we parameterize such an edge by ∞ (resp. 0) if it is connected to the terminal edge by an under-path (resp. an over-path). We parameterize the other edges in such a way that the parameters belong to $\mathbb{C} - \{0\}$, and satisfy the *hyperbolicity equations*



$$\Big(1 - \frac{x}{u}\Big)\Big(1 - \frac{v'}{x}\Big) = \Big(1 - \frac{x}{u'}\Big)\Big(1 - \frac{v}{x}\Big). \tag{2}$$

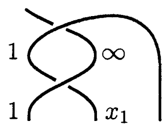
We consider a hyperbolic two-bridge knot K. Any open two-bridge knot can be presented by a plat closure of a 3-braid of a product of copies of σ_1 and σ_2^{-1}, *i.e.*, any open two-bridge knot diagram D (or its mirror image) can be obtained by gluing copies of the following tangle diagrams, which we call *elementary diagrams.*

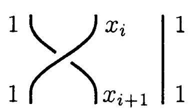


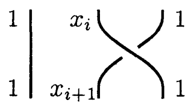
From the hyperbolicity equations among parameters of the resulting tangle diagram, the values of x_i are recursively determined by

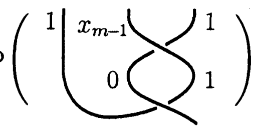
$$x_{i+1} = \begin{cases} x_i + 1 - \dfrac{x_i}{x_{i-1}} & \text{if the strand of } x_i \text{ is between } \sigma_1 \text{ and } \sigma_1 \\ & \qquad \text{or between } \sigma_2^{-1} \text{ and } \sigma_2^{-1}, \\ x_i + \dfrac{(x_i - 1)^2}{1 - \frac{x_i}{x_{i-1}}} & \text{otherwise.} \end{cases}$$

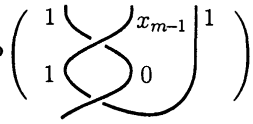
It is known that a hyperbolic structure of the complement of K is obtained from a parametrized diagram ([11], [13]). Calculating the monodromy representation, from the definition of $\tau(K)$, we can obtain a reformulation of $\tau(K)$. Explicitly, we define $\Phi(D)$ to be the composition of Φ of elementary diagrams whose values are given as follows,

$\Phi\Big($  $\Big)$ $= x_1(x_1 - 1)\begin{pmatrix} 1 & 2x_1 & 0 \end{pmatrix},$

$\Phi\Big($  $\Big)$ $= x_{i+1}\begin{pmatrix} 1 & 2\,x_{i+1} & 1 \\ 0 & -x_{i+1} & -1 \\ 0 & 0 & 1 \end{pmatrix},$

$\Phi\Big($  $\Big)$ $= x_{i+1}\begin{pmatrix} 1 & 0 & 0 \\ -1 & -x_{i+1} & 0 \\ 1 & 2\,x_{i+1} & 1 \end{pmatrix},$

$\Phi\Bigg($  $\Bigg) = \dfrac{x_{m-1}^3}{(x_{m-1}-1)^3}\begin{pmatrix}1\\-1\\2\end{pmatrix},$

$\Phi\Bigg($  $\Bigg) = \dfrac{x_{m-1}^3}{(x_{m-1}-1)^3}\begin{pmatrix}2\\-1\\1\end{pmatrix}.$

Then, we have that $\dfrac{2}{\tau(K)} = \Phi(D)$.

Let us review the definition of the Kashaev invariant. Let K be an oriented knot and $N \geq 2$. We put $q = \exp(2\pi\sqrt{-1}/N)$, $(x)_n = (1-x)(1-x^2)\cdots(1-x^n)$ and $\mathcal{N} = \{0, 1, \cdots, N-1\}$.

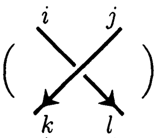
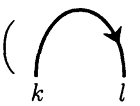
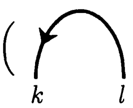
For $i, j, k, l \in \mathcal{N}$, we put

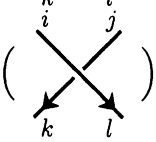
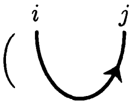
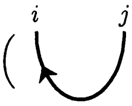
$$R^{ij}_{kl} = \frac{N\, q^{-\frac{1}{2}+i-k}\, \theta^{ij}_{kl}}{(q)_{[i-j]}(\overline{q})_{[j-l]}(q)_{[l-k-1]}(\overline{q})_{[k-i]}}, \qquad \overline{R}^{ij}_{kl} = \frac{N\, q^{\frac{1}{2}+j-l}\, \theta^{ij}_{kl}}{(\overline{q})_{[i-j]}(q)_{[j-l]}(\overline{q})_{[l-k-1]}(q)_{[k-i]}},$$

where $[m] \in \mathcal{N}$ is the residue of m modulo N, and we put

$$\theta^{ij}_{kl} = \begin{cases} 1 & \text{if } [i-j]+[j-l]+[l-k-1]+[k-i] = N-1, \\ 0 & \text{otherwise.} \end{cases}$$

Let D be an 1-tangle diagram of an open knot whose closure is the knot K. A *labeling* is an assignment of an element of $\mathcal{N}$ to each edge of D. We define the *weights* of labeled elementary tangle diagrams by

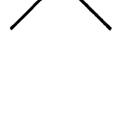
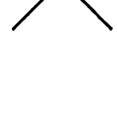
$W\Big($  $\Big) = R^{ij}_{kl}$, $W\Big($  $\Big) = q^{-1/2}\delta_{k,l-1}$, $W\Big($  $\Big) = \delta_{k,l}$,

$W\Big($  $\Big) = \overline{R}^{ij}_{kl}$, $W\Big($  $\Big) = q^{1/2}\delta_{i,j+1}$, $W\Big($  $\Big) = \delta_{i,j}$. (3)

Then, *the Kashaev invariant* $\langle K \rangle_N$ of K is defined by

$$\langle K \rangle_N = \sum_{\text{labelings}} \prod_{\substack{\text{crossings}\\ \text{of } D}} W(\text{crossings}) \prod_{\substack{\text{critical}\\ \text{points of } D}} W(\text{critical points}) \in \mathbb{C}.$$

We define the potential function for an open knot diagram parametrized by hyperbolicity parameters. We consider an angle consisting of two adjacent edges at a crossing, and associate such an angle with the value

 $\rightsquigarrow \ \mathrm{Li}_2(\frac{x}{y}) - \mathrm{Li}_2(1)$  $\rightsquigarrow \ \mathrm{Li}_2(1) - \mathrm{Li}_2(\frac{y}{x})$

where $\mathrm{Li}_2(z) = -\int_0^z \frac{\log(1-t)}{t}dt$. We define the potential function V to be the sum of such values for all angles except for the constant terms. We remark that the equations

$$\frac{\partial}{\partial x_i}V = 0 \qquad \text{for all } i$$

give the hyperbolicity equations and so, a solution of the hyperbolicity equations gives a critical point of V. Furthermore, it is known that

$$\log(q)_n \sim -\frac{N}{2\pi\sqrt{-1}}\mathrm{Li}_2(e^{2\pi\sqrt{-1}\frac{n}{N}}),$$

So, from the definition of the potential function, formally, we obtain the following approximation:

$$\langle K \rangle_N \sim \sum_{i_1,\ldots,i_m} \exp\left(\frac{N}{2\pi\sqrt{-1}}V(e^{2\pi\sqrt{-1}\frac{i_1}{N}},\ldots,e^{2\pi\sqrt{-1}\frac{i_m}{N}})\right).$$

Putting $\frac{i_1}{N} = t_1, \ldots, \frac{i_m}{N} = t_m$ and using the Poisson summation formula formally,

$$\langle K \rangle_N \sim N^m \int \exp\left(\frac{N}{2\pi\sqrt{-1}}V(e^{2\pi\sqrt{-1}t_1},\ldots,e^{2\pi\sqrt{-1}t_m})\right) dt_1\cdots dt_m.$$

Moreover, putting $x_i = e^{2\pi\sqrt{-1}t_i}$, we obtain

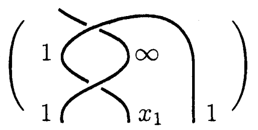
$$\langle K \rangle_N \sim N^m \int \exp\left(\frac{N}{2\pi\sqrt{-1}}V(x_1,\ldots,x_m)\right) dx_1\cdots dx_m.$$

By using the saddle point method formally and more calculations of the expansions, we obtain

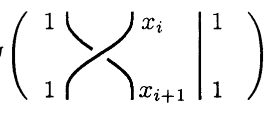
$$\langle K \rangle_N \sim e^{N\varsigma(K)} \cdot N^{3/2} \cdot \omega(K),$$

where $\varsigma(K) = \frac{1}{2\pi\sqrt{-1}}V(x_{1;c},\cdots,x_{m;c})$ for a critical point $(x_{1;c},\cdots,x_{m;c})$ of V and $\omega(K)$ can be written in terms of the Hessian of V at the critical point $(x_{1;c},\cdots,x_{m;c})$.

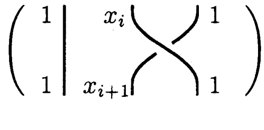
Moreover, we define $\Psi(D)$ to be the composition of Ψ of elementary diagrams whose values are given as follows,

Ψ 

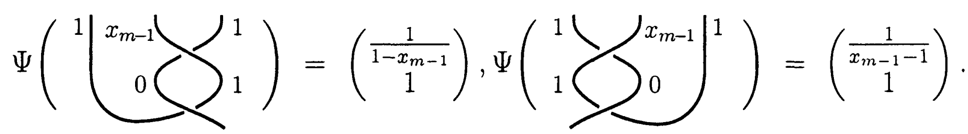
$= \begin{pmatrix} 1 & \frac{x_1}{1-x_1} \end{pmatrix},$

Ψ 

$= \frac{x_{i+1}}{x_i}\begin{pmatrix} -\frac{x_i(x_{i+1}-1)}{(x_i-1)x_{i+1}} & 1 \\ \frac{x_i-x_{i+1}}{x_{i+1}} & -\frac{x_i-1}{x_{i+1}-1} \end{pmatrix},$

Ψ 

$= \frac{x_{i+1}}{x_i}\begin{pmatrix} \frac{x_i(x_{i+1}-1)}{(x_i-1)x_{i+1}} & 1 \\ \frac{x_i-x_{i+1}}{x_{i+1}} & \frac{x_i-1}{x_{i+1}-1} \end{pmatrix},$

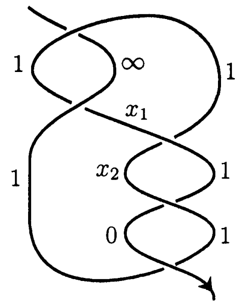


Noting that $\omega(K)^2$ can be presented in terms of the Hessian of the potential function defined from a parametrized open diagram, it follows that $\frac{1}{\sqrt{-1}\,\omega(K)^2} = \Psi(D)$.

Showing that the values of $\Phi(D)$ and $\Psi(D)$ satisfy the same recursion formula, we prove Theorem 1.

3 Example

In this section, we explain our results for the $\overline{5_2}$ knot K, which is presented by the following diagram D:



From (2), the hyperbolicity equations are presented by

$$(1-x_1)(1-\frac{1}{x_1}) = 1-\frac{x_2}{x_1}, \qquad (1-\frac{x_2}{x_1})(1-\frac{1}{x_2}) = 1-x_2.$$

Hence,

$$x_1^3 - 2\,x_1^2 + 3\,x_1 - 1 = 0.$$

Corresponding to the holonomy representation of the hyperbolic structure of the knot complement, we choose a solution

$$x_1 = 0.784920145... + \sqrt{-1}\cdot 1.307141278... \,,$$

which gives the complex hyperbolic volume by

$$\varsigma(K) = \frac{1}{2\pi\sqrt{-1}} V(x_1, x_2) = 0.450109610... - \sqrt{-1}\cdot 0.4813049796... \,.$$

Then, from the definition of $\Phi(D)$, we obtain

$$\frac{2}{\tau(K)} = x_1(x_1-1)\begin{pmatrix}1 & 2x_1 & 0\end{pmatrix}\cdot x_2\begin{pmatrix}1 & 0 & 0\\ -1 & -x_2 & 0\\ 1 & 2x_2 & 1\end{pmatrix}\cdot\frac{x_2^3}{(x_2-1)^3}\begin{pmatrix}1\\ -1\\ 2\end{pmatrix} \tag{4}$$

$$= -0.6323164993... + \sqrt{-1}\cdot 2.2345852998... \,, \tag{5}$$

and, hence, the value of the twisted Reidemeister torsion of K is give by

$$\tau(K) = -0.2344867659... - \sqrt{-1} \cdot 0.8286683659... \,. \tag{6}$$

Let us confirm that the above value is also obtained from [12], by transforming the Reidemeister torsion associated with the longitude (of [12]) to the Reidemeister torsion associated with the meridian (the above value) as mentioned in [5].

The knot group $\pi_1(K)$ of K is presented by $\pi_1(K) = \langle a, b \mid aw^2 = w^2 b\rangle$, where $w = ab^{-1}a^{-1}b$. The meridian longitude system (μ, λ) is presented in $\pi_1(K)$ by

$$\mu = a, \qquad \lambda = (ab^{-1}a^{-1}b)^2(ba^{-1}b^{-1}a)^2.$$

A non-abelian representation $\rho : \pi_1(K) \to \mathrm{SL}_2\mathbb{C}$ is parametrized by two parameters u and s as follows:

$$\rho(a) = \begin{pmatrix} \sqrt{s} & \frac{1}{\sqrt{s}} \\ 0 & \frac{1}{\sqrt{s}} \end{pmatrix}, \qquad \rho(b) = \begin{pmatrix} \sqrt{s} & 1 \\ -\sqrt{s}u & \frac{1}{\sqrt{s}} \end{pmatrix},$$

where s and u satisfies the Riley's equation $\phi_K(s,u) = 0$. The Riley's polynomial $\phi_K(s,u)$ [10] is given by

$$\phi_K(s,u) = -\frac{1}{s^2}(-2s+3s^2-2s^3+u-3su+6s^2u-3s^3u+s^4u-2su^2+3s^2u^2-2s^3u^2+s^2u^3).$$

The holonomy representation ρ_0 corresponds to the case $s = 1$ and $\phi_K(1,u) = 1 - 2u + u^2 - u^3$. By [12], the Reidemeister torsion $T_\lambda^{\rho_0}(K)$ associated with the longitude is given by

$$T_\lambda^{\rho_0}(K) = -\frac{(2+u)(2+7u)}{u^3(4+u^2)}.$$

Let $l_{1,1}(s,u)$ be the $(1,1)$-entry of $\rho(\lambda)$. As mentioned in [5], we can transform $T_\lambda^{\rho_0}(K)$ to the Reidemeister torsion $\tau(K)$ associated with the meridian by the formula

$$\pm\tau(K) = 2\left(\frac{\partial\, l_{1,1}}{\partial s} + \frac{\partial\, l_{1,1}}{\partial u}\frac{du}{ds}\right)\bigg|_{s=1} \frac{1}{T_\lambda^{\rho_0}(K)}.$$

Then, choosing the solution $u = 1 - x_1 = 0.21508 - \sqrt{-1}\cdot 1.30714$ of $\phi_K(1,u) = 0$ (see [8, Appendix D]), we obtain

$$\begin{aligned} \pm\tau(K) &= \frac{2u^4(2+u^2)(4+u^2)(2+4u^2+u^4)}{(2+u)(2+7u)} \\ &= -0.234487 - \sqrt{-1}\cdot 0.828668, \end{aligned}$$

which coincides with (6). Moreover, from the definition of $\Psi(D)$,

$$\begin{aligned} \frac{1}{\sqrt{-1}\,\omega(K)^2} &= \begin{pmatrix} 1 & \frac{x_1}{1-x_1} \end{pmatrix} \cdot \frac{x_2}{x_1} \cdot \begin{pmatrix} \frac{x_1(x_2-1)}{(x_1-1)x_2} & 1 \\ \frac{x_1-x_2}{x_2} & \frac{x_1-1}{x_2-1} \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{1-x_2} & 1 \end{pmatrix} \\ &= -0.632316... + \sqrt{-1}\cdot 2.23459..., \end{aligned}$$

which agrees with (5). On the other hand, in [6], it is rigorously shown that

$$\langle\, K \,\rangle_N \sim e^{N\varsigma(K)} \cdot N^{3/2} \cdot \omega(K). \tag{7}$$

Hence, we confirm Conjecture 2 for K.

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